Propagation of TE Modes in Nonuniform Waveguides*

H. ZUCKER†, MEMBER, IRE, AND G. I. COHN‡, SENIOR MEMBER, IRE

Summary—**The conditions for TE mode propagation in rectangular waveguides with nonuniform dielectric media are established. An equation is derived for determining the capacitivity functions which have equivalent curved waveguides with uniform dielectric media. The types of variable dielectric waveguides which have equivalent curved wall waveguides and a separable wave equation are determined.**

INTRODUCTION

APPLICATION of waveguides loaded with nonuniform dielectric material that varies in two dimensions has been limited by fabrication difficulties. The development of ferroelectric materials whose properties can be changed with the application of a biasing field considerably extends the practical applications to which nonuniformly loaded waveguides may be put. Such waveguides are also of intrinsic theoretical interest because of the mathematical techniques involved.

Solutions of the wave equations for propagation in nonuniform dielectric filled waveguides have been applied[1–5] to determine the propagating fields in uniformly loaded curved waveguides by using conformal coordinate transformations which transform curved waveguides into equivalent straight waveguides. The transformed wave equation is similar to the wave equation for propagation in a waveguide loaded with dielectric that varies in two dimensions.

This paper is part of a study concerned with propagation in nonuniformly loaded rectangular waveguides. The conditions for independent propagation of TE modes in such waveguides are determined. An equation is derived for determining which functional variations of the capacitivity have equivalent curved guides with uniform dielectric media. The types of equivalent waveguides which have a separable wave equation are also determined. Solutions for the propagating waves in nonuniformly-loaded straight-wall waveguide (where the nonuniformity is a function of one dimension only) are obtained.

A. Conditions for Independent Existence of Transverse Electric Modes in a Nonuniform Dielectric Loaded Waveguide

The problem to be treated, illustrated in Fig. 1, is a rectangular cross section cylindrical waveguide loaded with a dielectric that varies in the transverse-x and the longitudinal-z directions. The treatment is restricted to the case for which the walls of the waveguide and the dielectric have negligible loss.

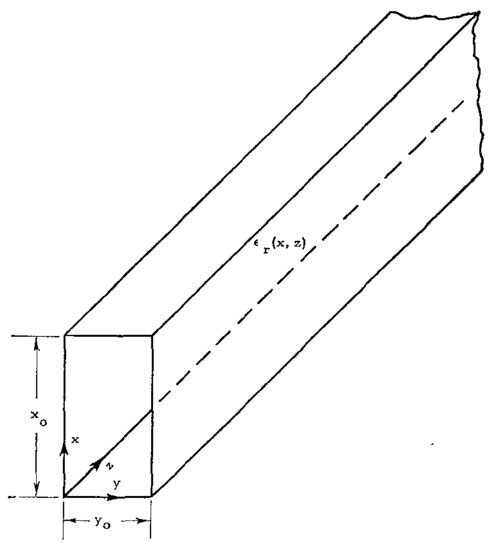

Fig. 1—Rectangular waveguide with dielectric that varies in the longitudinal direction and in one transverse direction.

In the absence of space charge and conduction current Maxwell's induction equations for isotropic media and for sinusoidal steady-state complex representation are

$$\nabla \times \boldsymbol{H} = j\omega\epsilon \boldsymbol{E} \quad (1)$$

$$\nabla \times \boldsymbol{E} = -j\omega\mu \boldsymbol{H} \quad (2)$$

where

$\boldsymbol{E}$ and $\boldsymbol{H}$ are the electric and magnetic field intensities,

μ and ϵ are the inductivity and capacitivity (may be functions of the three position coordinates x, y, z).

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† Armour Research Foundation, Chicago, Ill.
‡ Electro-Optical Systems, Inc., Pasadena, Calif.

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[4] R. Piloty, "The fields in rectangular waveguides of nonuniform cross-section excited with a TE_{10} wave," *Zeitschrift fur der Angewandte Physik*, vol. 1, pp. 490–502; November, 1949.

[5] H. Meinke, "Losungsverfahren fur inhomegene zylindersymmetrische Wellenfelder," *Zeitschrift fur der Angewandte Physik*, vol. 1, pp. 509–516; November, 1949.

$$\omega = 2\pi f \tag{3}$$

$$f = \text{frequency}.$$

Maxwell's Gaussian equations reduce to

$$\epsilon \nabla \cdot \boldsymbol{E} + \boldsymbol{E} \cdot \nabla \epsilon = 0 \tag{4}$$

$$\mu \nabla \cdot \boldsymbol{H} + \boldsymbol{H} \cdot \nabla \mu = 0. \tag{5}$$

The problem under consideration is restricted to homogeneous inductivity, hence (5) reduces to

$$\nabla \cdot \boldsymbol{H} = 0. \tag{6}$$

Obtaining separate equations for $\boldsymbol{E}$ and $\boldsymbol{H}$ from (1) and (2), gives

$$\nabla^2 \boldsymbol{H} + \left(\frac{\nabla \epsilon}{\epsilon}\right) \times (\nabla \times \boldsymbol{H}) + \beta_v^2 \epsilon_r \boldsymbol{H} = 0 \tag{7}$$

and

$$\nabla^2 \boldsymbol{E} - \nabla \nabla \cdot \boldsymbol{E} + \beta_v^2 \epsilon_r \boldsymbol{E} = 0 \tag{8}$$

where β_v is the propagation factor and

$$\beta_v = \omega \sqrt{\mu_v \epsilon_v} \tag{9}$$

and μ_v and ϵ_v are the inductivity and capacitivity of vacuum, ϵ_r is the relative capacitivity or dielectric constant.

The conditions for the TE mode propagation are established from (8). The z component of (8) is

$$\nabla^2 E_z - \partial_z(\nabla \cdot \boldsymbol{E}) + \beta_v^2 \epsilon_r E_z = 0. \tag{10}$$

For TE modes

$$E_z = 0. \tag{11}$$

This TE mode constraint reduces (10) to

$$\partial_z \nabla \cdot \boldsymbol{E} = 0. \tag{12}$$

Integrating (12) gives

$$\nabla \cdot \boldsymbol{E} = g(x, y) \tag{13}$$

where $g(x, y)$ is an arbitrary function of x and y.

For a uniformly loaded guide (4) reduces to

$$\nabla \cdot \boldsymbol{E}^u = \partial_x E_x^u + \partial_y E_y^u = 0. \tag{14}$$

At the junction between the uniform waveguide and the nonuniform dielectric loaded guide the electric fields E_x and E_y are the same on both sides of the boundary. If the fields are identical, so are their transverse derivatives. Therefore, at the junction, at $z = z_0$,

$$\partial_x E_x^i + \partial_y E_y^i = 0. \tag{15}$$

Since $g(x, y)$ does not vary with z and is zero at $z = z_0$, it is zero throughout the nonuniform dielectric region. Hence, for a nonuniformly loaded waveguide which is joined to a uniformly loaded waveguide and excited from that guide with a TE mode,

$$g(x, y) = 0. \tag{16}$$

For a TE mode, with $g(x, y)$ zero in (13), (4) reduces to

$$\boldsymbol{E} \cdot \nabla \epsilon = E_x \partial_x \epsilon_x + E_y \partial_y \epsilon_y = 0. \tag{17}$$

For all arbitrary variations of $\epsilon(x, y, z)$ (17) can only be satisfied if E_x and E_y are individually zero. Thus a TE mode cannot independently exist if the guide is filled with a dielectric that is an arbitrary function of position. Nontrivial solutions of (17) can exist only under the following conditions:

$$\begin{bmatrix} E_x \\ \partial_y \epsilon \end{bmatrix} = 0 \tag{18}$$

or

$$\begin{bmatrix} E_y \\ \partial_x \epsilon \end{bmatrix} = 0 \tag{19}$$

or

$$\begin{bmatrix} \partial_x \epsilon \\ \partial_y \epsilon \end{bmatrix} = 0. \tag{20}$$

Conditional equation sets (18) and (19) have already been derived for TE modes in waveguides where the dielectric constant varies in the transverse dimension only.[6]

Conditional equation set (20) is still satisfied if the capacitivity varies in the z dimension and according to (17) neither E_x and E_y has to be zero. TE_{mn} modes can therefore be independently present in a waveguide where the dielectric medium varies in the z dimension. The fields in such a waveguide can be obtained in terms of a product of three functions which are dependent on x, y and z respectively, with the functional dependence on x and y the same as for uniform waveguides and the z-dependent function can be obtained in terms of a second order differential equation with variable coefficients.

In the succeeding sections nonuniform dielectric loaded waveguides will be considered where the conditional equation set (18) is fulfilled. Under these conditions it follows from Maxwell's equations that

$$H_y = 0 \tag{21}$$

and from (15) and (18) that

$$\partial_y E_y = 0. \tag{22}$$

The conditions specified by (18), (21), and (22) rule out variation in the y direction; hence, only TE_{mo} modes can be present. With the above conditions the equation for $\boldsymbol{H}$, (7), has the y component equal to zero, and the equation for E_y, from (8) reduces to

$$\partial_x^2 E_y + \partial_z^2 E_y + \beta_v^2 \epsilon_r(x, z) E_y = 0. \tag{23}$$

[6] K. V. Malinowski and D. J. Angelokos, "Propagation in Inhomogeneously Filled Waveguides," University of California Institute of Engineering Research, Berkeley, Calif., Ser. No. 60, Issue No. 125; October, 1954.

The y component of the electric field satisfies (23), the simplest equation, of the above set. A solution for E_y, which satisfies (23) and the boundary conditions, is sufficient since the magnetic fields can be derived from it by differentiation.

B. Conformal Coordinate Transformation

The partial differential equation (23) can be expressed in the u, v-coordinate system by the transformation

$$u = u(x, z) \tag{24}$$

$$v = v(x, z) \tag{25}$$

where $u(x, z)$ and $v(x, z)$ are, for the time being, arbitrary functions.

Such a transformation generally converts the straight boundaries of the waveguide at the metal walls into curved boundaries. For an arbitrary transformation the resulting partial differential equation in the u and v coordinates would have a form more complex than (23). However, it is shown subsequently that for certain variations of the dielectric medium the transformation (if conformal) will transform the wave equation, (23), into a wave equation for a waveguide with uniform dielectric. The difference between the original and the transformed problem is that the boundaries and the dielectric have changed. The advantage of this method is that certain solutions for propagation in curved waveguides with uniform loading can be used as solutions to corresponding nonuniform dielectric loaded straight wall waveguides. This method is the inverse of the method used in solving certain waveguide problems with curved boundaries by transforming conformally the waveguide with curved boundaries into a waveguide with straight boundaries and a variable dielectric medium.[1–5]

In the new coordinate system the electric field E_y is a function of u and v.

$$E_y = E_y(u, v). \tag{26}$$

If u and v are chosen so that they satisfy the Cauchy-Riemann conditions

$$\partial_x u = \partial_z v \tag{27}$$

$$\partial_x v = -\partial_z u \tag{28}$$

then

$$(\partial_x u)^2 + (\partial_z u)^2 = (\partial_x v)^2 + (\partial_z v)^2 \tag{29}$$

$$\partial_x u \partial_x v + \partial_z u \partial_z v = 0 \tag{30}$$

$$\partial_x^2 u + \partial_z^2 u = 0 \tag{31}$$

$$\partial_x^2 v + \partial_z^2 v = 0. \tag{32}$$

Using (24) and (25) together with (27) through (32) transforms (23) to

$$(\partial_u^2 E_y + \partial_v^2 E_y)[(\partial_x u)^2 + (\partial_z u)^2] + \beta_v^2 \epsilon_r(x, z) E_y = 0. \tag{33}$$

If u satisfies the partial differential equation

$$(\partial_x u)^2 + (\partial_z u)^2 = \epsilon_r(x, z) \tag{34}$$

then the transformation (24) and (25) reduces (33) to

$$\partial_u^2 E_y + \partial_v^2 E_y + \beta_v^2 E_y = 0 \tag{35}$$

which is an equation for wave propagation in a vacuum filled waveguide. For $\beta_v = 0$, which corresponds to the static case, (23) is Laplace's equation and (35) obtained by conformal transformation, is also Laplace's equation.

Partial differential equation, (34), imposes a relationship between the coordinate u and the coordinates x and y. It follows from (29) that $v(x, z)$ also satisfies (34). The preliminary cases studied were chosen by picking a coordinate transformation and then finding the corresponding dielectric variation. For a specified dielectric variation the coordinate transformation is determined from (34). Eq. (34) is a nonlinear partial differential equation of the first order which can be reduced to a set of simultaneous ordinary differential equations of the first order. However, only specific variations of the dielectric medium $\epsilon(x, z)$ lead to a function which satisfies (34) and (31) simultaneously. These types of functions for the dielectric medium are derived in the subsequent section.

C. Partial Differential Equation for the Capacitivity Required in a Straight-Wall Waveguide having the Same Solution Uniformly Loaded Curved Wall Waveguide

The partial differential equation which $\epsilon_r(x, z)$ must satisfy if the nonuniform loaded straight wall waveguide is to have the same solution as the uniformly loaded curved wall waveguide can be obtained from (31) and (34).

Differentiating (34) once with respect to x gives

$$2\partial_x u \partial_x^2 u + 2\partial_z u \partial_{xz} u = \partial_x \epsilon_r \tag{36}$$

and once with respect to z gives

$$2\partial_x u \partial_{xz} u + 2\partial_z u \partial_z^2 u = \partial_z \epsilon_r. \tag{37}$$

Substituting (31) into (37) gives

$$2\partial_x u \partial_{xz} u - 2\partial_z u \partial_x^2 u = \partial_z \epsilon_r. \tag{38}$$

Eliminating $\partial_x^2 u$ between (36) and (38) gives

$$\partial_{xz} u = \frac{1}{2} \frac{\partial_x u \partial_z \epsilon_r + \partial_z u \partial_x \epsilon_r}{(\partial_x u)^2 + (\partial_z u)^2}. \tag{39}$$

Using (34) reduces (39) to

$$\partial_{xz} u = \frac{\partial_x u \partial_z \epsilon_r + \partial_z u \partial_x \epsilon_r}{2\epsilon_r}. \tag{40}$$

Substituting (40) into (36) and using (34) gives

$$\partial_x u = \frac{(\partial_x u \partial_x \epsilon_r - \partial_z u \partial_z \epsilon_r)}{2\epsilon_r}. \tag{41}$$

Differentiating (40) with respect to x gives

$$\partial_{x^2z}u = -\partial_x\epsilon_r\frac{(\partial_xu\partial_z\epsilon_r + \partial_zu\partial_x\epsilon_r)}{2\epsilon_r^2} + \frac{\partial_x^2u\partial_z\epsilon_r + \partial_xu\partial_{xz}\epsilon_r + \partial_{xz}u\partial_x\epsilon_r + \partial_zu\partial_x^2\epsilon_r}{2\epsilon_r} \quad (42)$$

and (41) with respect to z gives

$$\partial_{xz^2}u = -\partial_z\epsilon_r\frac{(\partial_xu\partial_x\epsilon_r - \partial_zu\partial_z\epsilon_r)}{2\epsilon_r^2} + \frac{\partial_{xz}u\partial_x\epsilon_r + \partial_xu\partial_{xz}\epsilon_r - \partial_zu\partial_z\epsilon_r - \partial_zu\partial_z^2\epsilon_r}{2\epsilon_r}. \quad (43)$$

Equating (42) and (43) and cancelling the same terms on both sides and using (31) to cancel terms which are identically zero gives

$$\frac{\partial_zu}{2\epsilon_r}\left[\partial_x^2\epsilon_r + \partial_z^2\epsilon_r - \frac{(\partial_x\epsilon_r)^2 + (\partial_z\epsilon_r)^2}{\epsilon_r}\right] = 0. \quad (44)$$

Cancelling $2\epsilon_r$ in (44), $\epsilon_r \neq 0$, leaves two factors which can be zero

$$\partial_x^2\epsilon_r + \partial_z^2\epsilon_r - \frac{(\partial_x\epsilon_r)^2 + (\partial_z\epsilon_r)^2}{\epsilon_r} = 0. \quad (45)$$

And

$$\partial_zu = 0. \quad (46)$$

However, (46) is a trivial solution since from (31)

$$\partial_x^2u = 0 \quad (47)$$

so ∂_xu is constant. Therefore from (34) $\epsilon_r(x, z)$ is constant. But this trivial solution is also contained in (45).

Examination of the wave equation, (24), shows that a uniformly filled waveguide with a capacitivity ϵ_r, can be treated in the u, v coordinate system as a uniform waveguide with the capacitivity of vacuum, but with the dimensions increased by the square of the relative capacitivity.

From (34) the general solution to (45) can be expressed in the form

$$\epsilon_r = (\partial_xu + j\partial_zu)(\partial_xu - j\partial_zu) \quad (48)$$

since

$$f(\zeta) = \frac{dw}{d\zeta} = \partial_xu + j\partial_xv = \partial_xu - j\partial_zu. \quad (49)$$

The general solution of (45) can be expressed as

$$\epsilon_r(x, z) = f(x + jz)f(x - jz) \quad (50)$$

where f is any complex analytic function.

The partial differential equation (45) has applications other than to the problem under consideration. According to (50), the square of the absolute value of any analytic complex function satisfies the partial differential equation (45). For a waveguide with a two dimensional variation of the dielectric medium (45) can serve as a criterion to determine if an equivalent curved wall waveguide with a uniform medium does exist.

D. Nonuniform Dielectric Loaded Straight Wall Waveguides Which Have an Equivalent Uniform Curved Wall Waveguide and a Separable Wave Equation

A relatively simple form of solution for wave propagation in waveguides with nonuniform dielectric media can be obtained if the wave equation (23) is separable. This is possible for dielectric media which have the form

$$\epsilon_r(x, z) = \epsilon_{rx}(x) + \epsilon_{rz}(z). \quad (51)$$

The solution for a dielectric function of this form reduces (23) to solving two ordinary second order differential equations with variable coefficients.

With the aid of (45) the types of variable dielectric waveguides that have a separable wave equation and an equivalent uniform curved wall waveguide can be determined. The number of such waveguides is limited and it has been shown[7] that uniform waveguides with curved walls coincident with coordinate surfaces of confocal conic section coordinate systems lead to a separable wave equation. This result is derived here based on (45) and constitutes an alternative derivation of the conditions for separability.

Substituting (51) into (45) gives

$$(\epsilon_{rx} + \epsilon_{rz})(\partial_x^2\epsilon_{rx} + \partial_z^2\epsilon_{rz}) - (\partial_x\epsilon_{rx})^2 - (\partial_x\epsilon_{rz})^2 = 0. \quad (52)$$

Differentiating (52) with respect to x and simplifying gives

$$(\epsilon_{rx} + \epsilon_{rz})\partial_x^3\epsilon_{rx} + \partial_x\epsilon_{rx}(\partial_z^2\epsilon_{rz} - \partial_x^2\epsilon_{rx}) = 0. \quad (53)$$

Differentiating (52) once with respect to z and simplifying gives

$$(\epsilon_{rx} + \epsilon_{rz})\partial_z^3\epsilon_{rz} + \partial_z\epsilon_{rz}(\partial_x^2\epsilon_{rx} - \partial_z^2\epsilon_{rz}) = 0. \quad (54)$$

Multiplying (53) by $\partial_z\epsilon_{rz}$ and (53) by $\partial_x\epsilon_{rx}$ and adding gives

$$(\epsilon_{rx} + \epsilon_{rz})(\partial_z\epsilon_{rz}\partial_x^3\epsilon_{rx} + \partial_x\epsilon_{rx}\partial_z^3\epsilon_{rz}) = 0. \quad (55)$$

Since the relative dielectric constant, $\epsilon_{rx} + \epsilon_{rz}$ is unequal to zero only the quantity in the brackets in (55) can be equal to zero. Eq. (55) separates into the two ordinary differential equations

$$\frac{d^3\epsilon_{rx}}{dx^3} + \delta^2\frac{d\epsilon_{rx}}{dx} = 0 \quad (56)$$

and

$$\frac{d^3\epsilon_{rz}}{dz^3} - \delta^2\frac{d\epsilon_{rz}}{dz} = 0 \quad (57)$$

[7] P. M. Morse and H. Feshback, "Methods of Theoretical Physics," McGraw-Hill Book Company, Inc., New York, N. Y., pp. 490–508; 1953.

where δ^2 is a real constant, and the partial differential operators have been replaced with the ordinary differential operators. Different solutions exist for (56) and (57) when $\delta=0$ and $\delta\neq 0$ and for δ^2 positive or negative. For $\delta=0$

$$\epsilon_{rx} = Ax^2 + Bx + C \tag{58}$$

$$\epsilon_{rz} = Dz^2 + Ez + F \tag{59}$$

where A, B, C, D, E and F are constant but are not independent since (58) and (59) have to also satisfy (52).

For $\delta\neq 0$ and δ^2 positive

$$x = A' \cos \delta x + B' \sin \delta x + C' \tag{60}$$

$$z = D' \cosh \delta z + E' \sinh \delta z + F' \tag{61}$$

where A', B', C', D', E', and F' are constants, but not independent.

For $\delta=0$ substituting (58) and (59) into (52) and comparing coefficients of the same power the following equation is obtained

$$\epsilon_r = \epsilon_{rx} + \epsilon_{rz} = A\left[\left(x + \frac{B}{2A}\right)^2 + \left(z + \frac{E}{2A}\right)^2\right]. \tag{62}$$

The coordinates of the equivalent uniform waveguide are found from

$$w = \sqrt{\frac{A}{4}}\,(\zeta + \zeta_1)^2 \tag{63}$$

where

$$\zeta_1 = \frac{B}{2a} + j\frac{E}{2A}\,. \tag{64}$$

The coordinates of the equivalent uniform waveguide are *parabolic.*

For $\delta\neq 0$ but positive proceeding in the same manner as for $\delta^2=0$ the following equation is obtained after some algebraic manipulations:

$$\epsilon_r = \frac{A}{\cos \delta x_1}\,[\cos \delta(x + x_1) + \cosh \delta(z + z_1)]. \tag{65}$$

The corresponding function w which gives the coordinates of the equivalent uniform waveguide is

$$w = \frac{2}{\delta}\sqrt{\frac{2A}{\cos \delta x_1}}\,\sin \frac{\delta}{2}\,(\zeta + \zeta_1) \tag{66}$$

where

$$\zeta_1 = x_1 + jz_1. \tag{67}$$

The corresponding coordinates are *elliptical* cylindrical. Setting

$$\tanh \delta z_1 = 1 \tag{68}$$

$$A\,\frac{\cosh \delta z_1}{\cos \delta x_1} = D \tag{69}$$

in (65) gives

$$\epsilon_r = De^{\delta z}. \tag{70}$$

The corresponding function w which gives the coordinates of the equivalent uniform waveguide

$$w = \frac{2\sqrt{D}}{\delta}\,e^{(\delta/2)\zeta}. \tag{71}$$

The coordinates for this case are *circular* cylindrical.

For $\delta^2\neq 0$ but δ^2 negative it follows from (56) and (57) and from the symmetry of (52) with respect to x and z that x and z can be interchanged in (65) and in (70). The corresponding equations to (65) and (66) are

$$\epsilon_r = \frac{A}{\cos \delta z_1}\,[\cos \delta(z + z_1) + \cosh \delta(x + x_1)] \tag{72}$$

and

$$w = \frac{2}{\delta}\sqrt{\frac{2A}{\cos \delta z_1}}\,\sin j\,\frac{\delta}{2}\,(\zeta + \zeta_1). \tag{73}$$

The coordinates are also *hyperbolical* cylindrical. The corresponding equations to (70) and (71) are

$$\epsilon_r = De^{\delta x} \tag{74}$$

and

$$w = 2\,\frac{\sqrt{D}}{\delta}\,e^{j\delta\zeta/2}. \tag{75}$$

These coordinates are also circular *cylindrical.*

E. Straight Waveguide Loaded with Dielectric that Varies in One Dimension and Their Equivalent Uniform Curved Wall Waveguides

The types of one dimensional variation of the capacitivity which lead to an equivalent curved wall waveguide with uniform loading are considered next. If the solution is known for the x variation it is also known for the z variation except for a constant of integration because of the symmetry of (45) in x and z. The relative capacitivity is taken as a function of x only in the problem treated, and hence the derivatives with respect to z vanish. Thus (45) reduces to

$$\partial_x{}^2\epsilon_r - \frac{1}{\epsilon_r}\,(\partial_x\epsilon_r)^2 = 0. \tag{76}$$

Using the transformation

$$y = \partial_x\epsilon_r = \frac{d}{dx}\,\epsilon_r \tag{77}$$

reduces (76) to

$$y\frac{dy}{d\epsilon} - \frac{1}{\epsilon}\,y^2 = y\left(\frac{dy}{d\epsilon_r} - \frac{1}{\epsilon_r}\,y\right) = 0. \tag{78}$$

One solution to (78) is $y=0$. From (77) this solution corresponds to $\epsilon_r=$constant which has been considered before. The other solution to (78) is

$$y = \alpha\epsilon_r. \tag{79}$$

Substituting (79) into (77) and solving the differential equation gives

$$\epsilon_r(x) = Ae^{\alpha x} \tag{80}$$

where A and α are constants.

From the symmetry of (45) a similar solution exists for capacitivity as a function of z only

$$\epsilon_r(z) = Be^{\beta z} \tag{80a}$$

where B and β are constants.

Shown here with the aid of (45) is that the one dimentional variations of the capacitivity given by (80) and (80a) are the only one dimensional variations which have an equivalent waveguide with a uniform dielectric constant, and curved walls.

These two cases are analyzed in detail to illustrate how the solution for the transformed waveguide is obtained from the solution for one type of straight waveguide.

1) Propagation in a Rectangular Waveguide Containing Dielectric with a Longitudinal Exponential Variation

To treat this problem let

$$w = \frac{2L}{\ln \epsilon_2} e^{-j(\ln \epsilon_2/2L)\zeta} \tag{81}$$

with

$$\zeta = x + jz. \tag{82}$$

From (81) and (34)

$$\epsilon_r(x, z) = \epsilon_2{}^{z/L}. \tag{83}$$

This section of the waveguide is transformed according to (81) into a waveguide in the u, v coordinate system that is a radial waveguide with a vacuum dielectric. The equivalent waveguides are shown in Fig. 2. The metal walls spaced x_0 apart in the x, z, coordinate system are transformed into radial walls enclosing the angle $2\phi_0$. From (81) and (82) the angle is given by

$$\phi_0 = \ln \epsilon_2 \frac{x_0}{4L} \cdot \tag{84}$$

The solution of the wave equation for the radial waveguide shown in Fig. 2(b) is known. The electric field E_y for the dominant TE_{10} mode is given by[8]

$$E_y = [AJ_v(\beta_v\rho) + BY_v(\beta_v\rho)] \cos\left(\frac{\pi\phi}{2\phi_0}\right) \tag{85}$$

[8] N. Marcuvitz, "Waveguide Handbook," McGraw-Hill Book Co., Inc., New York, N. Y., pp. 93–96; 1953.

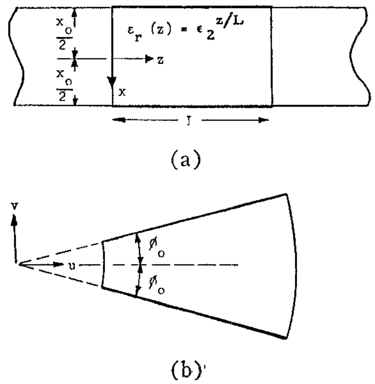

Fig. 2—(a) Rectangular waveguide with an exponentially varying dielectric in the longitudinal dimension. (b) Curved wall waveguide with uniform dielectric corresponding to waveguide in Fig. 2(a).

where ρ is the radial coordinate β_v is given by (9), A and B are constants, J_v and Y_v are Bessel functions of the first and second kind respectively of order

$$v = \frac{\pi}{2\phi_0} \cdot \tag{86}$$

From (81), and (83) it follows that

$$\rho = |w| = \frac{2L}{\ln \epsilon_2} \sqrt{\epsilon_r(z)} \tag{87}$$

$$\phi = \tfrac{1}{2} \ln \epsilon_2 x. \tag{88}$$

Substituting (84), (87), and (88) into (85) gives the electric field in the x, z coordinate system

$$E_y(x, z) = \left[AJ_v\left(\frac{2\beta_v L}{\ln \epsilon_2} \sqrt{\epsilon_r(z)}\right) + BY_v\left(\frac{2\beta_v L}{\ln \epsilon_2} \sqrt{\epsilon_r(z)}\right)\right] \cos\frac{x}{x_0} \cdot \tag{89}$$

It can be readily shown that (89) is a solution to the wave equation, (23), for the waveguide shown in Fig. 2(a).

2) Propagation in Rectangular Waveguide Containing Dielectric with a Transverse Exponential Variation

To treat this problem let

$$w = \frac{2x_0}{\ln \epsilon_2} e^{(\ln \epsilon_2/2x_0)\zeta}. \tag{90}$$

The corresponding capacitivity from (36) to

$$\epsilon_r(x) = \epsilon_2 \frac{x}{x_0} \cdot \tag{91}$$

The rectangular waveguide loaded with dielectric that has the exponential variation in the transverse dimension is transformed by (90) into a circular waveguide with metal walls at radii ρ_1 and ρ_2. The equivalent waveguides are shown in Fig. 3(a) and 3(b). From (90) the

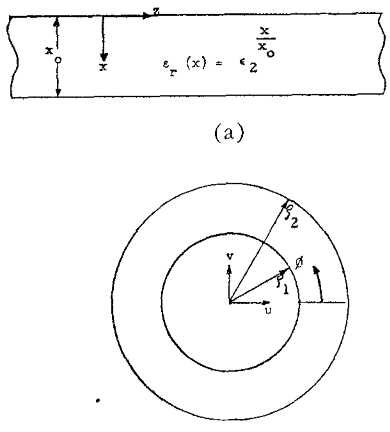

Fig. 3—(a) Rectangular waveguide with an exponentially varying dielectric in the transverse dimension. (b) Curved wall waveguide with uniform dielectric corresponding to waveguide in Fig. 3(a).

radii ρ_1 and ρ_2 are given by

$$\rho_1 = |w_1| = \frac{2x_0}{\ln \epsilon_2} \tag{92}$$

and

$$\rho_2 = |w_2| = \frac{2x_0\epsilon_2}{\ln \epsilon_2}\cdot \tag{93}$$

The angle ϕ in the circular waveguide determined from (90) is

$$\phi = \frac{\ln \epsilon_2}{2x_0} z. \tag{94}$$

As z varies from $-\infty$ to $+\infty$ the angle ϕ ranges over successive Riemann surfaces.

The wave equation in the u, v cylindrical coordinates is

$$\partial^2 E_y + \frac{1}{\rho} E_y + \frac{1}{\rho^2} \partial^2 \phi E_y + \beta_v^2 E_y = 0. \tag{95}$$

The solution to this equation for the transformed waveguide is

$$E_y = [J_{\beta_c}(\beta_v \rho) Y_{\beta_c}(\beta_v \rho_2) - Y_{\beta_c}(\beta_v \rho) J_{\beta_c}(\beta_v \rho_1)] \cdot [A e^{-\beta_c \phi} + B e^{\beta_c \phi}] \tag{96}$$

and the separation constant β_c is obtained from the conditional equation

$$J_{\beta_c}(\beta_v \rho_2) Y_{\beta_c}(\beta_v \rho_1) - Y_{\beta_c}(\beta_v \rho_2) J_{\beta_c}(\beta_v \rho_1) = 0. \tag{97}$$

Substituting the values for ρ, ρ_1, ρ_2 and ϕ from (90), (92), (93) and (94) into (96) and (97) gives the electric field and the conditional equation for the nonuniform dielectric waveguide. It can also be readily shown that (96) with the above substitutions is a solution to the wave equation (23) for a rectangular waveguide with an exponential variation of the dielectric in the transverse x dimension.

Conclusions

1) The necessary conditions for propagation of independent TE_{mn} modes in nonuniform dielectric waveguides is that the capacitivity varies in the longitudinal dimension only. Coupled TE_{mo} modes may propagate (without generating TM modes) if the capacitivity varies in only one transverse dimension and in the longitudinal dimension.

2) Certain curved wall waveguides with uniform media when transformed by a conformal coordinate transformation in straight wall waveguides with nonuniform media satisfy the same wave equation and vice versa. A derived partial differential equation, (45), for the capacitivity of a straight wall waveguide determines the existence of an equivalent curved wall waveguide with a uniform medium.

3) For the two-dimensional dielectric variations given by (51) the wave equation is separable and reduces to two ordinary second order differential equations for each independently propagating TE_{mo} mode. Some of these types of waveguides also have equivalent uniformly loaded curved wall waveguides. The curved walls are located at confocal conic section coordinate surfaces.